**Definition.** We say $a$ is an **absolute maximum (minimum)**, if for all $x$

$$f(x) \leq f(a)$$
$$(\geq)$$

**Theorem** (Extreme Value Theorem)**.** *If $f$ is continuous on $[a,b]$, then it has an absolute maximum and minimum.*

Before we prove EVT, lets first show the weaker result.

**Lemma.** *If $f$ is continuous on $[a,b]$, then it is bounded.*

*Proof.* Let us assume that $f$ is unbounded form above on $[a,b]$, and we will show there is a contradiction. We proceed by bisection. We define,

$$L_0, R_0] = [a,b].$$

Note that since $f$ is unbounded on $[L_0, R_0]$, we must have it is bounded on of the halves of $[L_0, R_0]$. Let $[L_1, R_1]$ be that half and pick a point $P_1$ such that

$$f(P_1) > 1.$$

Now since $f$ is unbounded on $[L_1, R_1]$, we must have it is bounded on of the halves of $[L_1, R_1]$. Let $[L_2, R_2]$ be that half and pick a point $P_2$ such that

$$f(P_2) > 2.$$

Repeating in this we get

$$[a,b] = [L_0, R_0] \supset [L_1, R_1] \supset [L_2, R_2] \supset \cdots \supset [L_n, R_n] \supset \ldots$$

Also we get a sequence of points $P_n \in [L_n, R_n]$ such that

$$f(P_n) > n.$$

Since the lengths of the intervals is going to 0 (Why?), we have have there is a $c$ such that

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = c.$$

Also because $L_n \leq P_n \leq R_n$, we have

$$\lim_{n\to\infty} P_n = c.$$

Since $f$ is continuous at $c$, there is a $\delta > 0$ such, $c - \delta < x < c + \delta$ implies

$$f(c) - < f(x) < f(c) + 1.$$

Because $P_n$ converges to $c$, we have there is a $N > f(c) + 1$ such that

$$c - \delta < P_N < c + \delta.$$

Thus we have $f(P_N) < f(c) + 1$ and $f(P_n) > N > f(c) + 1$. Which is a contradiction. So $f$ is bounded from above. An analogous argument shows $f$ is bounded from below. □

*Proof of Extreme value theorem.* Let us show that $f$ has a n absolute maximum. By the lemma we know that $f$ is bounded. Let $U$ be the least upper bound for $f$. in $[a, b]$. In other words we pick $U$ to the smallest number such that,

$$f(x) \leq U \quad x \in [a, b].$$

$f$ will have an absolute maximum is there is a $x_0$ such that $f(x_0) = U$. Let's assume no such $x_0$ exists, so for all $x$ we have $f(x) < U$ or, $U - f(x) > 0$. Since $f$ is continuous, so is $U - f(x)$, and since it's strictly positive, we have,

$$g(x) = \frac{1}{U - f(x)}$$

is continuous on $[a, b]$. Again by the lemma, we have $g$ is bounded above by some $V$, so for all $x \in [a, b]$,

$$\frac{1}{U - f(x)} \leq V.$$

By rearranging we get,

$$f(x) \leq U - \frac{1}{V}.$$

So $f$ is bounded by $U - \frac{1}{V} < U$, which contradicts the fact that $U$ was the smallest upper bound. Thus we must have $f$ has an absolute maximum.

A nearly identical proof shows that $f$ has an absolute minimum, and is left as an exercise to the reader. □